NOTE FOR UNIQUENESS OF SOLUTIONS OF NAVIER-STOKES IN 2D

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In this note, we give an alternative proof of solutions of the Navier-Stokes equations in $\mathcal{C}([0,T), L^{2,1}(\mathbb{R}^2))$ based on the atomic decomposition of the Lorentz space. The first proof can be found in [3].

The Navier-Stokes equations read

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \Delta u \\ \nabla \cdot u = 0 \\ u|_{t=0} = u_0. \end{cases} \qquad x \in \mathbb{R}^2, t \in \mathbb{R}_+ \tag{0.1}$$

Here $u\colon \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}^2$ is the velocity field and the scalar fields $p\colon \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}$ denote the pressure. The integral formulation of the Navier-Stokes equations read:

$$\begin{aligned} u(t) &= e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot (u \otimes u)(s) ds \\ &= e^{t\Delta}u_0 - \int_0^t \int_{\mathbb{R}^2} K_{t-s}(x-y)(u \otimes u)(s,y) dy ds = e^{t\Delta}u_0 + B(u,u) \end{aligned}$$

with $K_{t-s}(x-y)$ satisfies

$$|K_{t-s}(x-y)| \lesssim \frac{1}{|t-s|^{3/2} + |x-y|^3} \lesssim \min\left\{ \frac{1}{|t-s|^{1-\delta}|x-y|^{1+2\delta}}, \frac{1}{|t-s|^{1+\delta}|x-y|^{1-2\delta}} \right\}$$

for a positive δ.

Theorem 1. *If $u, v \in \mathcal{C}([0,T), L^{2,1})$ are two solutions of the Cauchy problem* (0.1) *with the same initial data, then $u = v$.*

In our proof, we will use the atomic decomposition of Lorentz space $L^{p,q}$ (see Tao's lecture note). Let us state it as follows.

Lemma 2. *Let $0 < p < \infty$. Then any $f \in L^{p,q}$ can be written as*

$$f = \sum_{k=-\infty}^{\infty} c_k \chi_k,$$

where each χ_k is a function bounded by $O(2^{-\frac{k}{p}})$ and supported on a set of measure $O(2^k)$, and the c_k are non-negative constants such that $\|f\|_{L^{p,q}} \sim \|c_k\|_{l^q}$

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The proof of Theorem 1 follows the idea of [1], [3] and [4]. Therefore, it suffices to prove the following proposition .

Proposition 3. *The bilinear map* $B(u,v): L_t^\infty L_x^{2,1}\times L_t^\infty L_x^{2,\infty}\to L_t^\infty L_x^{2,\infty}$ *is bounded.*

Proof. We shall prove

$$\|B(u,v)\|_{L_t^\infty L_x^{2,\infty}}\lesssim \|u\|_{L_t^\infty L_x^{2,1}}\|v\|_{L_t^\infty L_x^{2,\infty}}.$$

By duality, the inequality above is in turn equivalent to prove the following trilinear form estimate:

$$|T(u,v,w)|\lesssim \|u\|_{L_t^\infty L_x^{2,1}}\|v\|_{L_t^\infty L_x^{2,\infty}}\|w\|_{L^1 L_x^{2,1}}$$

where $T(u,v,w):=\int_0^T\int_{\mathbb{R}^2}\int_{s<t}\int_{\mathbb{R}^2}K_{t-s}(x-y)(u\otimes v)(s,y)w(t,x)dydsdxdt$. By decomposing $T(u,v,w)$ dyadically as $\sum_j T_j(u,v,w)$, where the summation is over the integers $\mathbb{Z}$ and

$$T_j(u,v,w)=\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{|t-s|\sim 2^{-j}}K_{t-s}(x-y)(u\otimes v)(s,y)w(t,x)dsdydxdt.$$

So it suffices to prove the estimate

$$\sum_j|T_j(u,v,w)|\lesssim \|u\|_{L_t^\infty L_x^{2,1}}\|v\|_{L_t^\infty L_x^{2,\infty}}\|w\|_{L^1 L_x^{2,1}}.$$

By applying Lemma 1, we have the decomposition (see [2])

$$u(t,x)=\sum_m a_m(t)\chi_{E_m}(t),\; v(t,x)=\sum_n b_n(t)\lambda_{F_n}(t),\; w(t,x)=\sum_k c_k(t)\sigma_{G_k}(t)$$

where for each t,m, the function $\chi_{E_m}(t)$ is bounded by $O(2^{-\frac{m}{2}})$ and supported on a set E_k of measure $O(2^k)$, and similarly for $\lambda_{F_n}(t)$ and $\sigma_{G_n}(t)$. Moreover, the functions $a_m(t)$, $b_n(t)$ and $c_k(t)$ satisfy

$$\|a_m\|_{l^1}\lesssim\|u\|_{L^{2,1}},\qquad \|b_n\|_{l^\infty}\lesssim\|v\|_{L^{2,\infty}},\qquad \|c_k\|_{l^1}\lesssim\|w\|_{L^{2,1}}.$$

Then, for each $T_j(u,v,w)$,

$$\begin{aligned}
|T_j(u,v,w)|&\lesssim\int_0^T\int_{|t-s|\sim 2^{-j}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{|t-s|^{3/2}+|x-y|^3}u(s,y)v(s,y)w(t,x)\\
&\lesssim\int_0^T\int_{|t-s|\sim 2^{-j}}\sum_{m,n,k}\int_{E_m(s)\cap F_n(s)}\int_{G_k(t)}\frac{2^{-\frac{m+n+k}{2}}}{|t-s|^{3/2}+|x-y|^3}|a_m(s)||b_n(s)||c_k(t)|\\
&\lesssim\int_0^T\int_{|t-s|\sim 2^{-j}}\sum_{m,n,k}|a_m(s)||b_n(s)||c_k(t)|\\
&\quad\int_{E_m(s)\cap F_n(s)}\int_{G_k(t)}2^{-\frac{m+n+k}{2}}\min\left\{\frac{1}{|t-s|^{1-\delta}|x-y|^{1+2\delta}},\frac{1}{|t-s|^{1+\delta}|x-y|^{1-2\delta}}\right\}
\end{aligned}$$

For fixed s,t, by rearrangement, we have

$$\int_{E_m(s)\cap F_n(s)}\int_{G_k(t)}\frac{1}{|x-y|^{1+2\delta}}dydx\lesssim\int_{E_m(s)\cap F_n(s)}\int_{B(0,\sqrt{\frac{|G_k(t)|}{\pi}})}\frac{1}{|z|^{1+2\delta}}dzdy\lesssim 2^{k(\frac{1}{2}-\delta)}\min\{2^m,2^n\}$$

and

$$\int_{E_m(s)\cap F_n(s)}\int_{G_k(t)}\frac{1}{|x-y|^{1-2\delta}}dydx \lesssim \int_{E_m(s)\cap F_n(s)}\int_{B(0,\sqrt{\frac{|G_k(t)|}{\pi}})}\frac{1}{|z|^{1-2\delta}}dzdy \lesssim 2^{k(\frac{1}{2}+\delta)}\min\{2^m,2^n\}$$

Thus, we have

$$\begin{aligned}|T_j(u,v,w)| \lesssim &\int_0^T\int_{|t-s|\sim 2^{-j}}\sum_{m,n,k}|a_m(s)||b_n(s)||c_k(t)| \\ &\min\{2^{j(1-\delta)-k\delta},2^{j(1+\delta)+k\delta}\}2^{-\frac{m+n}{2}}\min\{2^m,2^n\} \\ \lesssim &\int_0^T\int_{|t-s|\sim 2^{-j}}\sum_{m,n,k}|a_m(s)||b_n(s)||c_k(t)|\min\{2^{-(j+k)\delta},2^{(j+k)\delta}\}2^{j-\frac{|m-n|}{2}} \\ &\int_0^T\int_{|t-s|\sim 2^{-j}}\sum_{m,n,k}|a_m(s)||b_n(s)||c_k(t)|2^{-|j+k|\delta}2^{j-\frac{|m-n|}{2}}\end{aligned}$$

Summing in j we have

$$\begin{aligned}\sum_j|T_j(u,v,w)| &\lesssim \int_0^T\sum_{j,k}\int_{|t-s|\sim 2^{-j}}2^j2^{-|j+k|\delta}|c_k(t)|\sum_{m,n}|a_m(s)||b_n(s)|2^{-\frac{|m-n|}{2}}dsdt \\ &\lesssim \int_0^T\sum_{j,k}\int_{|t-s|\sim 2^{-j}}2^j2^{-|j+k|\delta}|c_k(t)|\|a_m(s)\|_{l^1}\|b_n(s)\|_{l^\infty}dsdt \\ &\lesssim \int_0^T\sum_{j,k}2^{-|j+k|\delta}|c_k(t)|dt\|u\|_{L^\infty L^{2,1}}\|v\|_{L^\infty L^{2,\infty}} \\ &\lesssim \int_0^T\|w\|_{L^{2,1}}\|u\|_{L^\infty L^{2,1}}\|v\|_{L^\infty L^{2,\infty}} \\ &\lesssim \|u\|_{L^\infty L^{2,1}}\|v\|_{L^\infty L^{2,\infty}}\|w\|_{L^1L^{2,1}}\end{aligned}$$

This ends the proof. □

References

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